RESOLVENT CONVERGENCE OF STURM-LIOUVILLE OPERATORS WITH SINGULAR POTENTIALS

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Abstract. In this paper we consider the Sturm-Liuoville operator in the Hilbert space L_2 with the singular complex potential of W_2^{-1} and two-point boundary conditions. For this operator we give sufficient conditions for norm resolvent approximation by the operators of the same class.

1. Main result

Let on a compact interval $[a,b]$ the formal differential expression

$$l(y) = -y''(t) + q'(t)y(t), \qquad q(\cdot) \in L_2([a,b],\mathbb{C}) =: L_2. \tag{1}$$

be given.

This expression can be defined as the Shin-Zettl [1] quasi-differential expression with following quasi-derivatives [2]:

$$D^{[0]}y = y, \quad D^{[1]}y = y' - qy, \quad D^{[2]}y = -(D^{[1]}y)' - qD^{[1]}y - q^2 y.$$

In this paper we consider the set of quasi-differential expressions $l_\varepsilon(\cdot)$ of the form (1) with potentials $q_\varepsilon(\cdot) \in L_2$, $\varepsilon \in [0,\varepsilon_0]$. In the Hilbert space L_2 with norm $\|\cdot\|_2$ each of these expressions generates a dense closed quasi-differential operator $L_\varepsilon y := l_\varepsilon(y)$,

$$Dom(L_\varepsilon) := \{y \in L_2 : \exists D_\varepsilon^{[2]} y \in L_2; \quad \alpha(\varepsilon)\mathcal{Y}_a(\varepsilon) + \beta(\varepsilon)\mathcal{Y}_b(\varepsilon) = 0\},$$

where matrices $\alpha(\varepsilon), \beta(\varepsilon) \in \mathbb{C}^{2\times 2}$, and vectors

$$\mathcal{Y}_a(\varepsilon) := \{y(a), D_\varepsilon^{[1]}y(a)\}, \quad \mathcal{Y}_b(\varepsilon) := \{y(b), D_\varepsilon^{[1]}y(b)\} \in \mathbb{C}^2.$$

Recall that operators L_ε converge to L_0 in the sense of norm resolvent convergence, $L_\varepsilon \xrightarrow{R} L_0$, if there exists a number $\mu \in \mathbb{C}$ such that $\mu \in \rho(L_0)$ and $\mu \in \rho(L_\varepsilon)$ (for all sufficiently small ε) and

$$\|(L_\varepsilon - \mu)^{-1} - (L_0 - \mu)^{-1}\| \to 0, \quad \varepsilon \to +0.$$

This definition does not depend on the point $\mu \in \rho(L_0)$ [3].

For the case where matrices $\alpha(\varepsilon), \beta(\varepsilon)$ do not depend on ε, paper [2] gives following

Theorem 1. *Suppose $\|q_\varepsilon - q_0\|_2 \to 0$ for $\varepsilon \to +0$ and the resolvent set of the operator L_0 is not empty. Then $L_\varepsilon \xrightarrow{R} L_0$.*

Our goal is to generalize Theorem 1 onto the case of boundary conditions depending on ε and to weaken conditions on potentials applying results of papers [4, 5].

Denote by $c^\vee(t) := \int_a^t c(x)dx$ and by $\|\cdot\|_C$ the sup-norm.

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Theorem 2. *Suppose the resolvent set of the operator L_0 is not empty and for $\varepsilon \to +0$:*

1) $\|q_\varepsilon\|_2 = O(1)$;
2) $\|(q_\varepsilon - q_0)^\vee\|_C \to 0$;
3) $\|(q_\varepsilon^2 - q_0^2)^\vee\|_C \to 0$;
4) $\alpha(\varepsilon) \longrightarrow \alpha(0), \quad \beta(\varepsilon) \longrightarrow \beta(0)$.

Then $L_\varepsilon \overset{R}{\to} L_0$.

Note that condition 3) is not additive.

Condition 1) (taking into account 2), 3)) may be weakened in several directions.

Actually we will prove a stronger statement on the considered operators' Green functions' convergence with respect to the norm $\|\cdot\|_\infty$ of the space L_∞ on the square $[a,b]\times[a,b]$.

2. Comparison of Theorems 1 and 2

We are going to show that if $\|q_\varepsilon - q_0\|_2 \to 0$, $\varepsilon \to +0$, then conditions 1), 2), 3) of Theorem 2 are true.

Indeed, $\|q_\varepsilon\|_2 \le \|q_\varepsilon - q_0\|_2 + \|q_0\|_2 = O(1)$.

Also

$$|\int_a^t (q_\varepsilon - q_0)ds| \le \int_a^b |q_\varepsilon - q_0|ds \le (\int_a^b |q_\varepsilon - q_0|^2 ds)^{1/2}(b-a)^{1/2} \to 0,\ \varepsilon \to +0.$$

$$|\int_a^t (q_\varepsilon^2 - q_0^2)ds| \le \int_a^b |q_\varepsilon^2 - q_0^2|ds \le \int_a^b |q_\varepsilon - q_0||q_\varepsilon + q_0|ds \le$$

$$\le (\int_a^b |q_\varepsilon - q_0|^2 ds)^{1/2}(\int_a^b |q_\varepsilon + q_0|^2 ds)^{1/2} \to 0, \quad \varepsilon \to +0.$$

Following example proves Theorem 2 to be stronger than Theorem 1.

Example 1. Suppose $q_0(t) \equiv 0$, $q_\varepsilon(t) = e^{it/\varepsilon}$, $t \in [0,1]$.

The set of operators L_ε defined by these potentials does not satisfy assumptions of Theorem 1 because

$$\|q_\varepsilon - q_0\|_2^2 = \|q_\varepsilon\|_2^2 = \int_0^1 |q_\varepsilon|^2 ds \equiv 1.$$

It is evident that functions $q_\varepsilon(\cdot)$ do not converge to 0 even with respect to the Lebesgue measure. However, they satisfy conditions 1), 2), 3) of Theorem 2. Indeed, $\|q_\varepsilon\|_2 \le 1$. Moreover,

$$\|q_\varepsilon^\vee\|_C = \|\int_0^t e^{is/\varepsilon} ds\|_C \le 2\varepsilon \to 0, \quad \varepsilon \to +0.$$

$$\|(q_\varepsilon^2)^\vee\|_C = \|\int_0^t (e^{is/\varepsilon})^2 ds\|_C \le \varepsilon \to 0, \quad \varepsilon \to +0.$$

3. Preliminary result

Consider a boundary-value problem

$$y'(t;\varepsilon) = A(t;\varepsilon)y(t;\varepsilon) + f(t;\varepsilon), \quad t \in [a,b], \quad \varepsilon \in [0,\varepsilon_0] \tag{3.1_ε}$$

$$U_\varepsilon y(\cdot;\varepsilon) = 0, \tag{3.2_ε}$$

where matrix functions $A(\cdot,\varepsilon) \in L_1^{m\times m}$, vector-functions $f(\cdot,\varepsilon) \in L_1^m$, and linear continuous operators $U_\varepsilon : C([a,b];\mathbb{C}^m) \to \mathbb{C}^m$.

We recall from [4, 5]

Definition Denote by $\mathcal{M}^m[a,b] =: \mathcal{M}^m$, $m \in \mathbb{N}$ the class of matrix functions $R(\cdot;\varepsilon) : [0,\varepsilon_0] \to L_1^{m\times m}$, such that the solution of the Cauchy problem

$$Z'(t;\varepsilon) = R(t;\varepsilon)Z(t;\varepsilon), \quad Z(a;\varepsilon) = I_m$$

satisfies the limit condition

$$\lim_{\varepsilon\to+0} \|Z(\cdot;\varepsilon) - I_m\|_C = 0.$$

Sufficient conditions for $R(\cdot;\varepsilon) \in \mathcal{M}^m$ derive from [6]. To prove Theorem 2 we apply the simplest of them

$$\|R(\cdot;\varepsilon)\|_1 = O(1), \quad \|R^\vee(\cdot;\varepsilon)\|_C \to 0,$$

where $\|\cdot\|_1$ is the norm in $L_1^{m\times m}$.

Paper [5] gives the following general

Theorem 3. *Suppose*

1) *the homogeneous limit boundary-value problem* $(3.1_0), (3.2_0)$ *with* $f(\cdot;0) \equiv 0$ *has only zero solution;*
2) $A(\cdot;\varepsilon) - A(\cdot;0) \in \mathcal{M}^m$;
3) $\|U_\varepsilon - U_0\| \to 0, \quad \varepsilon \to +0.$

Then for sufficiently small ε *Green matrices* $G(t,s;\varepsilon)$ *of problems* (3.1_ε), (3.2_ε) *exist and on the square* $[a,b]\times[a,b]$

$$\|G(\cdot,\cdot;\varepsilon) - G(\cdot,\cdot;0)\|_\infty \to 0, \quad \varepsilon \to +0. \tag{4}$$

Condition 3) of Theorem 3 cannot be replaced by a weaker condition of the strong convergence of the operators $U_\varepsilon \xrightarrow{s} U_0$ [5]. However, one may easily see that for multi-point "boundary" operators

$$U_\varepsilon y := \sum_{k=1}^{n} B_k(\varepsilon)y(t_k), \quad \{t_k\} \subset [a,b], \quad B_k(\varepsilon) \in \mathbb{C}^{m\times m}, \quad n \in \mathbb{N},$$

both conditions of strong and norm convergence are equivalent to

$$\|B_k(\varepsilon) - B_k(0)\| \to 0, \quad \varepsilon \to +0, \quad k \in \{1,...,n\}.$$

4. Proof of Theorem 2

We give two lemmas to apply Theorem 3 to proof of Theorem 2.

Lemma 1. *Function* $y(t)$ *is a solution of a boundary-value problem*

$$D_\varepsilon^{[2]}y(t) = f(t;\varepsilon) \in L_2, \quad \varepsilon \in [0,\varepsilon_0], \tag{5}$$

$$\alpha(\varepsilon)\mathcal{Y}_a(\varepsilon) + \beta(\varepsilon)\mathcal{Y}_b(\varepsilon) = 0. \tag{6}$$

if and only if vector-function $w(t) = (y(t), D_\varepsilon^{[1]}y(t))$ *is a solution of a boundary-value problem*

$$w'(t) = A(t;\varepsilon)w(t) + \varphi(t;\varepsilon), \tag{7}$$

$$\alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0, \tag{8}$$

where matrix function

$$A(\cdot;\varepsilon) := \begin{pmatrix} q_\varepsilon & 1 \\ -q_\varepsilon^2 & -q_\varepsilon \end{pmatrix} \in L_1^{2\times 2}, \tag{9}$$

and $\varphi(\cdot;\varepsilon) := (0, -f(\cdot;\varepsilon))$.

Proof. Consider the system of equations

$$\begin{cases} (D_\varepsilon^{[0]}y(t))' = q_\varepsilon(t)D_\varepsilon^{[0]}y(t) + D_\varepsilon^{[1]}y(t) \\ (D_\varepsilon^{[1]}y(t))' = -q_\varepsilon^2(t)D_\varepsilon^{[0]}y(t) - q_\varepsilon(t)D_\varepsilon^{[1]}y(t) - f(t;\varepsilon) \end{cases}$$

If $y(\cdot)$ is a solution of equation (5), then definition of quasi-derivatives derives that $y(\cdot)$ is a solution of this system. On the other hand with

$$w(t) = (D_\varepsilon^{[0]}y(t), D_\varepsilon^{[1]}y(t)) \qquad \text{and} \qquad \varphi(t;\varepsilon) = (0, -f(t;\varepsilon))$$

this system may be rewritten in the form of equation (7).

As $\mathcal{Y}_a(\varepsilon) = w(a)$, $\mathcal{Y}_b(\varepsilon) = w(b)$ then it is evident that boundary conditions (6) are equivalent to boundary conditions (8).

Lemma 2. *Let the assumption*

($\mathcal{E}$) *Homogeneous boundary-value problem* $D_0^{[2]}y(t) = 0, \quad \alpha(0)\mathcal{Y}_a(0) + \beta(0)\mathcal{Y}_b(0) = 0$ *has only zero solution*

be fulfilled. Then for sufficiently small ε *Green function* $\Gamma(t,s;\varepsilon)$ *of the semi-homogeneous boundary problem (5), (6) exists and*

$$\Gamma(t,s;\varepsilon) = -g_{12}(t,s;\varepsilon) \qquad a.\ e.,$$

where $g_{12}(t,s;\varepsilon)$ *is the corresponding element of the Green's matrix*

$$G(t,s;\varepsilon) = (g_{ij}(t,s;\varepsilon))_{i,j=1}^2$$

of two-point vector boundary-value problem (7), (8).

Proof. Taking into account Theorem 3 and Lemma 1 assumption ($\mathcal{E}$) derives that homogeneous boundary-value problem

$$w'(t) = A(t;\varepsilon)w(t), \quad \alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0$$

for sufficiently small ε has only zero solution.

Then for problem (7), (8) Green matrix

$$G(t,s,\varepsilon) = (g_{ij}(t,s))_{i,j=1}^2 \in L_\infty^{2\times 2}$$

exists and the unique solution of (7), (8) is written in the form

$$w_\varepsilon(t) = \int_a^b G(t,s;\varepsilon)\varphi(s;\varepsilon)ds, \quad t \in [a,b], \quad \varphi(\cdot;\varepsilon) \in L_2.$$

The last equality can be written in the form

$$\begin{cases} D_\varepsilon^{[0]}y_\varepsilon(t) = \int_a^b g_{12}(t,s;\varepsilon)(-\varphi(s;\varepsilon))ds \\ D_\varepsilon^{[1]}y_\varepsilon(t) = \int_a^b g_{22}(t,s;\varepsilon)(-\varphi(s;\varepsilon))ds, \end{cases}$$

where $y_\varepsilon(\cdot)$ is the unique solution of problem (5), (6). This implies the assertion of Lemma 2.

Now, passing to the proof of Theorem 2, we note that since

$$(q_\varepsilon + \mu)^2 - (q_0 + \mu)^2 = (q_\varepsilon^2 - q_0^2) + 2\mu(q_\varepsilon - q_0),$$

in view of conditions 2), 3) we can assume without loss of generality that $0 \in \rho(L_0)$. Let's prove that

$$\sup_{\|f\|_2=1} \|L_\varepsilon^{-1} f - L_0^{-1} f\| \to 0, \quad \varepsilon \to +0.$$

Equation $L_\varepsilon^{-1} f = y_\varepsilon$ is equivalent to the relation $L_\varepsilon y_\varepsilon = f$, that is y_ε is the solution of the problem (5), (6) and due to inclusion $0 \in \rho(L_0)$ the assumption $(\mathcal{E})$ of Lemma 2 holds. Conditions 1)–3) of Theorem 2 imply that $A(\cdot;\varepsilon) - A(\cdot;0) \in \mathcal{M}^2$, where $A(\cdot;\varepsilon)$ is given by (9). Therefore assumption of Theorem 2 derives that assumption of Theorem 3 for problem (7), (8) is fulfilled. This means that Green matrices $G(t,s;\varepsilon)$ of the problems (7), (8) exist and limit relation (4) holds. Taking into account Lemma 2, this implies the limit equality

$$\|\Gamma(\cdot,\cdot;\varepsilon) - \Gamma(\cdot,\cdot;0)\|_\infty \to 0, \quad \varepsilon \to +0.$$

Then

$$\begin{aligned}\|L_\varepsilon^{-1} - L_0^{-1}\| = \sup_{\|f\|_2=1} \| \textstyle\int_a^b [\Gamma(t,s;\varepsilon) - \Gamma(t,s;0)] f(s) ds\|_2 \le \\ (b-a)^{1/2} \sup_{\|f\|_2=1} \| \textstyle\int_a^b |\Gamma(t,s;\varepsilon) - \Gamma(t,s;0)||f(s)|ds\|_C \le \\ (b-a)\|\Gamma(\cdot,\cdot;\varepsilon) - \Gamma(\cdot,\cdot;0)\|_\infty \to 0, \quad \varepsilon \to +0,\end{aligned}$$

which implies the assertion of Theorem 2.

5. Three extensions of Theorem 2

As was already noted, the assumptions of Theorem 2 may be weakened. Let

$$R(\cdot;\varepsilon) := A(\cdot;\varepsilon) - A(\cdot;0)$$

where $A(\cdot;\varepsilon)$ is given by (9).

Theorem 4. *In the statement of Theorem 2, condition 1) can be replaced by any one of the following three more general (in view of 2) and 3)) asymptotic conditions as $\varepsilon \to +0$:*
(I) $\|R(\cdot;\varepsilon) R^\vee(\cdot;\varepsilon)\|_1 \to 0$;
(II) $\|R^\vee(\cdot;\varepsilon) R(\cdot;\varepsilon)\|_1 \to 0$;
(III) $\|R(\cdot;\varepsilon) R^\vee(\cdot;\varepsilon) - R^\vee(\cdot;\varepsilon) R(\cdot;\varepsilon)\|_1 \to 0$.

Proof. The proof of Theorem 4 is similar to the proof of Theorem 2 with following remark to be made. Condition 2) of Theorem 3 holds if (see [6]) $\|R^\vee(\cdot;\varepsilon)\|_C \to 0$ and either the condition $R(\cdot;\varepsilon)\|_1 = O(1)$ (as in Theorem 2), or any of three conditions (I), (II), (III) of Theorem 4 holds.

Following example shows each part of Theorem 4 to be stronger than Theorem 2.

Example 2. Let $q_0(t) \equiv 0$, $q_\varepsilon(t) = \rho(\varepsilon) e^{it/\varepsilon}$, $t \in [0,1]$.

One may easily calculate that conditions

$$\rho(\varepsilon) \uparrow \infty, \quad \varepsilon \rho^3(\varepsilon) \to 0, \quad \varepsilon \to +0,$$

imply assumptions 2), 3) of Theorem 2 and any one of assumptions (I), (II), (III) of Theorem 4. But assumption 1) of Theorem 2, does not hold because $\|q_\varepsilon - q_0\|_2 \uparrow \infty$.

For Schrödinger operators of the form (1) on $\mathbb{R}$ with real-valued periodic potential q', where $q \in L_2^{loc}$, self-adjointness and sufficient conditions for norm resolvent convergence were established in [7]. For other problems related to those studied in [2], see also [8], [9].

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